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Scaling and Critical Phenomena in a Cellular Automaton Slider-Block Model for Earthquakes

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The dynamics of a general class of two-dimensional cellular automaton slider-block models of earthquake faults is studied as a function of the failure rules that determine slip and the nature of the failure threshold. Scaling properties of clusters of failed sites imply the existence of a mean-field spinodal line in systems with spatially random failure thresholds, whereas spatially uniform failure thresholds produce behavior reminiscent of self-organized critical behavior. This model can describe several classes of faults, ranging from those that only exhibit creep to those that produce large events.

KEY WORDS: Earthquakes; faults; spinodal; nucleation; scaling; critical phenomena.

Earthquakes are physical processes exhibiting a wealth of complex phenomena, including space-time clustering of events,[1] scaling,[2] and migration of activity along fault systems.[1,3] A number of years ago, Burridge and Knopoff[4] (BK) proposed a model useful in understanding some of these observations. The BK model consists of a network of blocks coupled by springs with force constants K_C sliding on a frictional surface. If the magnitude of the force vector ζ_i on block i is increased to the point where it exceeds a prescribed threshold value ζ_i^F, the block slides or jumps a distance U_i in the direction of the force, thereby reducing the force on that block to a residual value ζ^R. Each block that slides may induce failure of neighboring blocks by means of the coupling through the springs, leading to clusters of failed blocks. In the classical BK model, massive blocks were used, together with a particular form of velocity-weakening friction, thereby leading to a series of nonlinear differential equations to be

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solved for the block positions. A cellular automaton version of this model was subsequently introduced[5] in which the blocks are taken to be massless, and simple jump rules are introduced using elementary concepts of "static" and "dynamic" friction.

The dynamics of these models has been the subject of considerable interest, initially among seismologists[4,5] and more recently in the condensed matter community.[6] For the most part the latter work has studied the behavior of massive blocks subject to a velocity-weakening friction force and has emphasized the similarities with the self-organized criticality (SOC) model of sandpiles.[7] However, there is considerable evidence[6,8,9] that the SOC paradigm is not rich enough to describe all the phenomena observed in earthquakes. In particular, the existence of limit cycles with the appearance of earthquakes of a characteristic size for a given fault system and the existence of great events as well as creep that cannot be described with scaling indicate that SOC is at best a partial description of the earthquake phenomenon.

For example, there exist segments of fault displaying both a scaling range of minor events up to some small limiting value, and events in which the entire segment fails as a unit. These large events are called "characteristic earthquakes." Examples include the Parkfield section of the San Andreas fault and the segment of subduction zone that failed in the great 1964 Prince William Sound, Alaska, earthquake.[2] We argue that the minor events may be analogous to spinodal fluctuations, whereas the characteristic earthquakes are analogous to nucleation events. Events analogous to first-order transitions, for which evidence exists in real sandpile experiments[10] as well, are not predicted by SOC theories.

In this work we address these problems by considering a slider-block model that focuses attention on the influence of the failure mechanism and the external loading. Our main result is that the statistical nature of the observed phenomena is strongly influenced by the loading, the form of the failure threshold, and the mechanism for energy dissipation. In particular we find that failure thresholds that have a spatially constant value apparently leads to phenomena that appear to be SOC for short times, while spatially varying thresholds, whether random, fractal, or some other configuration, lead to the existence of critical points that bear a strong resemblance to mean-field spinodals. This dependence is in better qualitative agreement with the data on earthquakes, which show much more varied behavior than that analogous to SOC.

In our slider-block model the displacement U_i at a time $t+1$ is given by

$$U_i(t+1) = U_i(t) + J[\zeta_i(t)]\, \Theta[\zeta_i(t) - \zeta_i^{\mathrm{F}}] \tag{1}$$

where Θ is the Heaviside function and $J[\zeta_i(t)]$ specifies the size of the jump if $\zeta_i > \zeta_i^{\mathrm{F}}$. The force on each block is

$$\zeta_i(t) = \sum_{ij} T_{ij} U_j + Q_i(t) \tag{2}$$

where T_{ij} includes both self-interactions $[T_{ii}]$ and nearest neighbor terms. The term $Q_i(t)$ specifies the external driving force. This model is quite general and is similar in spirit to one described by Feder and Feder.[6]

We are concerned with a system driven continually toward failure by the applied force $Q_i(t)$ corresponding to a loader spring with a force constant K_L. The loader spring couples each block to a plate that is displaced forward an amount $V\Delta$ at fixed time intervals $\Delta \gg 1$, thereby stretching the spring. Consequently,

$$Q_i(t) = K_L V\Delta \left[\sum_n \Theta[t - n\Delta] \right] \tag{3}$$

In this work we will restrict our consideration to systems in $d = 2$ and adopt the values $T_{ii} = -[K_L + 4K_C] = -5$ and $T_{ij} = K_C = 1$ for i and j nearest neighbors. Clearly $\sum_j T_{ij} = -K_L$. We use the jump function $J[\zeta_i(t)] = [\zeta_i(t) - \zeta^{\mathrm{R}}]/K$, where $K = K_L + 4K_C$.

The evolution of the system is as follows: We move the loader plate a distance $V\Delta$ and calculate the force on each block. Those blocks that have $\zeta_i > \zeta_i^{\mathrm{F}}$ are then moved an amount $J[\zeta_i]$ and the forces on each block recalculated. Blocks that have $\zeta_i > \zeta_i^{\mathrm{F}}$ are again moved and the forces recalculated. This process is repeated until no block has a force that exceeds the threshold. At this point the loader plate is moved again and the process repeats. We analyze the properties of clusters of failed sites. A cluster is defined as a group of nearest neighbor sites that fail during updating following a single increment of the loader plate. Each block is counted only once irrespective of the number of times it failed during the updating following a single loader plate move. All simulations carried out in this paper were started from random initial block positions. Data were taken only after several tens of thousands of loader updates had occurred (typically a half million clusters), to eliminate effects associated with system transients. Blocks on the boundaries of the network were connected to the loader plate as well as to their three (edge blocks) or two (corner blocks) nearest neighbors. Other boundary conditions are possible, including periodic or fixed edge blocks.

First we considered the model with a flat (spatially uniform) failure threshold $\zeta_i^{\mathrm{F}} = \zeta^{\mathrm{F}}$ for all i. We measured the density of clusters with s failure sites, $sn(s)$, as a function of the "velocity" V. For $V = 0.1$ and $V = 10$ we

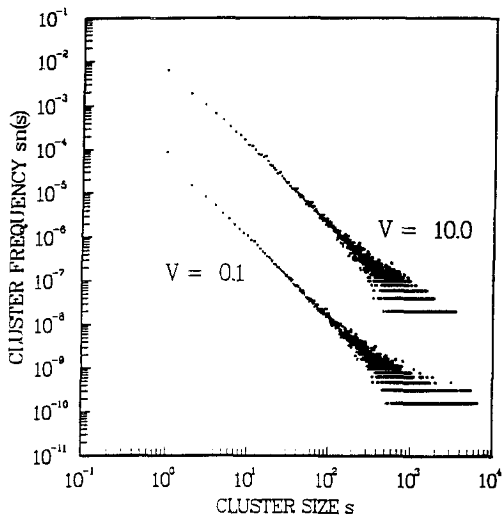

Fig. 1. Log–log plot of the density of clusters of size s, $sn(s)$, vs. s for two velocities in a system with a flat failure threshold. The slope is consistent with a $\tau \sim 2$. The lack of dependence of the slopes on V indicates SOC.

found (Fig. 1) that $sn(s) \sim s^{1-\tau}$ with $\tau \sim 2$. It appears from these data that for the constant failure threshold, $sn(s)$ is independent of V, i.e., the system seems to exhibit SOC. We have also performed simulations of the same model with $K_C = 25$, $K_L = 1$ and found SOC-like behavior with $\tau \sim 2.5$ which is mean field.[11] In this regard $(K_C/K_L)^{1/2}$ appears to play the role of the interaction range used in thermal models of phase transitions.[12] If our interpretation is correct, the correlation length is $\xi \sim (K_C/K_L)^{1/2} (V_c - V)^{-\nu}$.

In our second set of runs for this model we altered the failure threshold. The reason for this change is that the surfaces of faults are rough, and can only be in intermittant contact. A more accurate representation of a fault therefore is a spatially nonuniform threshold. We used both a threshold in which the ζ_i^F were generated at random at each site and a fractal threshold in which the ζ_i^F are thought of as the z coordinates of the points on a surface, measured from the x–y plane, with a fractal dimension of 2.2. The results for both thresholds were similar (Fig. 2). The data of Fig. 2 taken with the random threshold clearly show a strong nonlinear regime for $V = 8$. As the velocity increases the plot becomes more linear, i.e., critical. It appears that the random failure threshold has altered the apparent SOC behavior obtained with flat thresholds to one with a critical point at $V_c = 19$. The slope of the line at V_c indicates that the exponent $\tau \sim 2.5$. The fractal failure threshold gave a similar value for τ with $V_c = 14$.

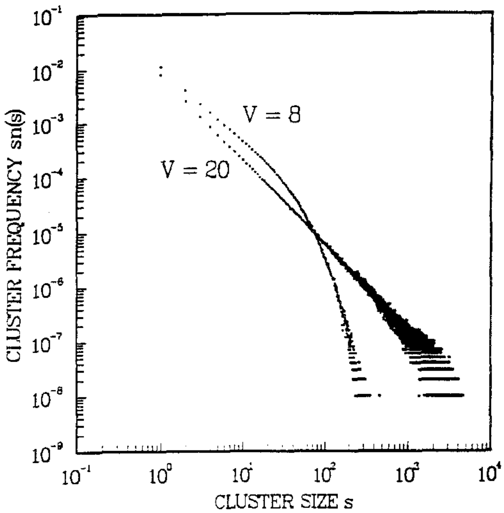

Fig. 2. Log–log plot of $sn(s)$ vs. s for a system with a failure threshold that is a random number uniformly distributed between 50 and 350. The velocity dependence indicates critical phenomena rather than SOC. The slope of the curve at $V = V_c = 8$, which was obtained from 3.0×10^6 clusters, is $\tau - 1 \sim 1.4$, and at $V = V_c = 20$, using 1.9×10^6 clusters, is $\tau - 1 \sim 1.5$.

In order to understand this phenomenon, it is first useful to note that the value of $\tau = 2.5$ is characteristic of mean-field percolation and spinodals.[12] Our interpretation of these results is that the critical point in the random threshold model is a spinodal. As the randomness of the failure threshold is reduced, the line of spinodal singularities approaches an SOC point. In particular, the threshold randomness is a relevant scaling field with respect to the SOC line, whereas the "velocity" V is the relevant scaling field with respect to the spinodal line.

On a microscopic scale the random nature of the threshold appears to limit the size of most clusters grown from the initial failed sites or seeds. At the velocities studied in this work the density of seeds is small compared to the random site percolation threshold, hence infinite clusters are formed via a coalescence of several smaller clusters which are themselves composed of many blocks. However, coalescence is rare due to the fact that individual cluster perimeters are composed of blocks that have recently failed and hence have a small stress near the residual value. Moreover, the clusters are bordered by blocks that have not failed and hence have, in general, a high failure threshold. These conditions combine to make the probability of two clusters coalescing at a particular site rare. Due to the individual cluster size, however, the number of potential coalescing or binding sites is large. This morphology is similar to that of the gelation of large, branched

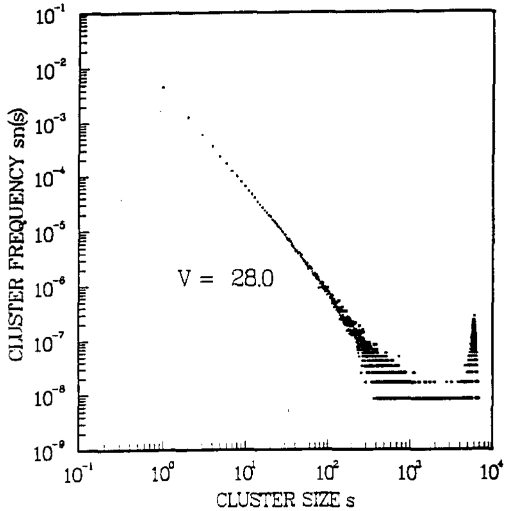

Fig. 3. Log–log plot of $sn(s)$ vs. s for a system with the same failure threshold as Fig. 2. The measurement was made for 10^6 clusters. The velocity is considerably higher than V_c and "infinite clusters" frequently occur.

polymers,[13] which also has mean-field percolation exponents. To test this hypothesis, we have measured the number of initiator sites $n_I(s)$ in clusters of size s and found[14] that $n_I(s)/s \to 0$ as $s \to \infty$.

As further evidence for the interpretation of the critical point as a spinodal we find that for velocities larger than the critical velocity V_c there is an instability similar to continuous ordering in thermal phase transitions[15] and for $V < V_c$ we have an indication of nucleation phenomena. Thus the velocity acts as a scaling field. Moreover, measurements (Fig. 4) indicate that correlation functions have the correct asymptotic form,[13] $e^{-R/\xi}/R^{d-2+\eta}$, with $\eta \sim 0$. Data obtained for the dependence of the correlation length on the scaling field $V_c - V$ are less convincing, but seem to approach power laws with $\nu \sim 0.5$ as expected. These results will be discussed in a future publication.[14] It is also important to note that the phenomena seen in this model are strongly dependent on the velocity. For the limit $V \to 0$, where there is only one initiator,[6] there is evidently no spinodal singularity.

Finally, we have found that for fixed V, increasing K_C/K_L produces a transition in model behavior from "macroscopic stable sliding" to "macroscopic stick slip." Behavior of this type occurs in laboratory experiments[16] as machine stiffness K_m is decreased. Assuming that K_m in the laboratory plays the same role as K_L in the model, this effect is then evidently a

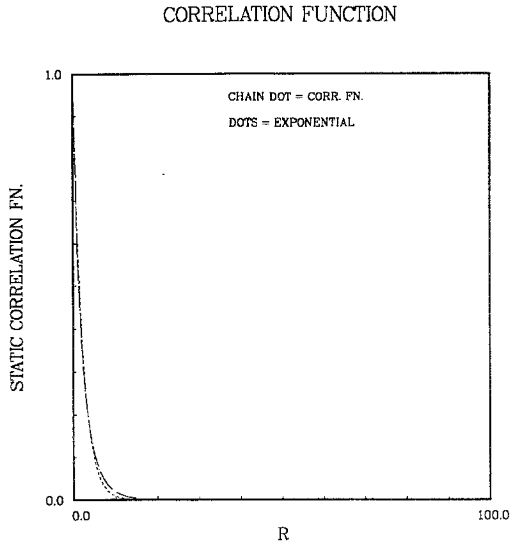

Fig. 4. Plot of the static correlation function $C(R)$ against R for $V=8$. The chain-dotted line is the correlation function, the dashed line is $e^{-R/2.1}$.

consequence of correlation length $\xi \sim (K_C/K_L)^{1/2}\,(V_c - V)^{-\nu}$ growing to a magnitude that exceeds the system size L.

In summary: We have investigated a cellular automaton slider-block model with a spatially varying failure threshold that is a considerably more realistic model for an earthquake fault than the spatially uniform threshold models which appear to exhibit SOC. In contrast to the spatially uniform threshold, we obtain critical rather than SOC-like phenomena. As the dispersion in the failure threshold decreases, we apparently cross over to the SOC-like regime.[14] This result is significant for understanding earthquake faults, as it allows a variety of behavior, depending on the dispersion of the failure thresholds and the plate velocity relative to the critical value. Due to the existence of a spinodal and the implied metastability, there should exist rare nucleation events that are quite large. These events have been seen.[14] Finally, the dispersion in the failure threshold brings the velocity into play as a relevant scaling field.

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